

Investigation of Bose condensation in ideal Bose gas trapped under generic power law potential in d dimension

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Abstract

The changes in characteristics of Bose condensation of ideal Bose gas due to an external generic power law potential $U = \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i}$ are studied carefully. Detailed calculation of Kim *et al.* (S. H. Kim, C. K. Kim and K. Nahm, J Phys. Condens. Matter 11 10269 (1999).) yielded the hierarchy of condensation transitions with changing fractional dimensionality. In this manuscript, some theorems regarding specific heat at constant volume C_V are presented. Careful examination of these theorems reveal the existence of hidden hierarchy of the condensation transition in trapped systems as well.

1 Introduction

Many authors have investigated the thermodynamic properties of Bose gas[1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 13], particularly after it was possible to create Bose Einstein Condensation (BEC) in magnetically trapped alkali-metal gases[20, 21, 22]. The constrained role of external potential does change the characteristics of quantum gases[14, 15, 16, 17, 18, 19], providing an exciting opportunity to study the quantum mechanical effects. It is seen in the studies that the thermodynamic behavior of non-relativistic quantum gases are governed by polylogarithm functions both in the case of trapped[10, 14, 15, 16] and free[23, 24, 25, 26] quantum gases. It is also reported that the polylogarithm functions can give a single unified picture of bosons and fermions for free[23] and trapped[28] quantum gases as well. Both in the case of Bose and Fermi gases, different structural properties of polylogarithms[9, 15, 27, 29] are related to different statistical effects. The trapping potential radically changes different thermodynamic properties of quantum gases[15, 14, 27, 28]. Also the behavior of these quantities do change with dimensionality. Hence, it will be very intriguing to study the properties of Bose gas in detail before and after the critical temperature in arbitrary dimension with generic trapping potential of the form, $U = \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i}$. In this report, we give a special emphasis on specific heat at constant volume C_V as well as its derivatives, which are the salient features to understand BEC[30].

In a previous study with free Bose gas in arbitrary dimension, Kim *et al.*[30] reported that the dimensionality contribution is a dominant factor to specify the behavior of BEC[30]. In general, there exists no discontinuity in C_V at $T = T_C$ for free Bose gas in three dimensional space. However, its derivative is discontinuous where the magnitude of the discontinuity is finite ($\frac{3.665 Nk}{T_C}$). Nevertheless, this is not true for free Bose gas in any arbitrary dimension. For instance, when $d > 4$, C_V is itself discontinuous at $T = T_C$ [5] for free Bose gas. In the case of trapped Bose gas with harmonic potential[5, 14], the scenario completely changes as C_V becomes discontinuous even in $d = 3$. In order to understand the critical behavior of free Bose system more closely, Kim *et al.* performed a calculation to check the discontinuity of l -th derivative of C_V at $T = T_C$, which yields the “class” of the C_V function changes with dimensionality. This calculation shows that there exists a hierarchy of condensation transitions with changing fractional dimensionality. It is well known that, there is no BEC for free Bose gas in $d \leq 2$ and Kim *et al.* found C_V to be smooth function in the whole temperature range in this situation. If the dimensionality ranges from $2 < d \leq 3$, the discontinuity of l -th derivative of C_V at $T = T_C$ depends upon the sub-interval we choose in this range (see Table 1 of Ref. [30]), while the first derivative of C_V remains discontinuous for $d > 3$. In this report, we have proposed a new theorem, concerning the l -th derivative of C_V for Bose gas trapped under generic power law potential, which coincides with the calculation of Kim *et al.*, in the limit all $n_i \rightarrow \infty$. Therefore one can reproduce the previous calculation as a special case of the new theorem. This new theorem helps

us to understand the hidden hierarchy of condensation transition, existing in trapped systems as well. Most importantly, one can find situations even in an integer dimension where the l -th derivative of C_V of trapped Bose gas is continuous, with appropriate trapping potential.

In this work, the grand potential is determined at first, from which all the thermodynamic quantities such as internal energy E , entropy S , Helmholtz free energy F and C_V are calculated. In order to scrutinize them closely, the fugacity (chemical potential) is evaluated numerically in arbitrary dimension with any trapping potential as a function of reduced temperature $\tau = \frac{T}{T_C}$ (see appendix) following Kelly's work[31]. Then we propose the theorems regarding C_V and its derivative. Beside this, the latent heat of condensation transition is studied in detail from the Clausius-Clapayron equation[6], from which we have deduced the required latent heat for Bose condensation in arbitrary dimension with any trapping potential. All the calculated quantities reduce to text book result of free Bose gas when all $n_i \rightarrow \infty$ [5].

The report is organized in the following way. The grand potential and the other thermodynamic quantities are presented in section 2 and 3, respectively. In section 4 we deduce C_V and the significant theorems regarding this. Section 5 is devoted to investigate the latent heat of condensation for trapped system. Results are discussed in section 6. The report is concluded in section 7.

2 Density of States and grand potential

Considering the ideal Bose gas in a confining external potential in a d -dimensional space with Hamiltonian,

$$\epsilon(p, x_i) = bp^l + \sum_{i=1}^d c_i \left| \frac{x_i}{a_i} \right|^{n_i} \quad (1)$$

Where, p is the momentum and x_i is the i th component of coordinate of a particle and b, l, a_i, c_i, n_i are all positive constants. Here, c_i, a_i and n_i determine the depth and confinement power of the potential. Using $l = 2, b = \frac{1}{2m}$ one can get the energy spectrum of the Hamiltonian used in the literature [5, 6, 11, 15, 9]. Note that, $x_i < a_i$. For the free system all $n_i \rightarrow \infty$. As $|\frac{x_i}{a_i}| < 1$, the potential term goes to zero when all $n_i \rightarrow \infty$.

The resulting density of states with this Hamiltonian is[9],

$$\rho(E) = C(m, V'_d) E^{x-1} \quad (2)$$

where, $C(m, V'_d)$ is a constant depending on effective volume V'_d and mass m [9, 14]

The grand potential for the Bose system,

$$q = - \sum_{\epsilon} \ln(1 - z \exp(-\beta \epsilon)) \quad (3)$$

where k, μ and $z = \exp(\beta \mu)$ being the Boltzmann Constant, the chemical potential and fugacity respectively and $\beta = \frac{1}{kT}$.

Replacing the sum by integration we get,

$$q = q_0 - \int_0^{\infty} \rho(\epsilon) \ln(1 - z \exp(-\beta \epsilon)) \quad (4)$$

which yields,[9, 14]

$$q = q_0 + g \frac{V'_d}{\lambda^d} g_{x+1}(z) \quad (5)$$

Where,

$$g_l(z) = \int_0^\infty dx \frac{x^{l-1}}{z^{-1}e^x - 1} = \sum_{j=1}^\infty \frac{z^j}{j^l} \quad (6)$$

$$q_0 = -\ln(1-z) \quad (7)$$

$$V'_d = V_d \prod_{i=1}^d \left(\frac{kT}{c_i}\right)^{1/n_i} \Gamma\left(\frac{1}{n_i} + 1\right) \quad (8)$$

$$\lambda' = \frac{hb^{\frac{1}{l}}}{\pi^{\frac{1}{2}}(kT)^{\frac{1}{l}}} \left[\frac{d/2+1}{d/l+1}\right]^{1/d} \quad (9)$$

$$\chi = \frac{d}{l} + \sum_{i=1}^d \frac{1}{n_i} \quad (10)$$

And, V'_d and $g_\chi(z)$ is known as effective volume and Bose function respectively. Now, as $z \rightarrow 1$, the Bose function $g_\chi(z)$ approaches $\zeta(\chi)$, for $\chi > 1$ [5]. Another representation of Bose function $g_\chi(z)$ due to Robinson is [5, 30]

$$g_\chi(e^{-\eta}) = \frac{\Gamma(1-\chi)}{\eta^{1-\chi}} + \sum_{i=0}^\infty \zeta(\chi-i)\eta^i \quad (11)$$

3 Thermodynamic quantities

The number of particles can be evaluated from the grand potential[9, 14],

$$\begin{aligned} N &= z\left(\frac{\partial q}{\partial z}\right)_{\beta, V} \\ \Rightarrow N - N_0 &= g \frac{V'_d}{\chi^d} g_\chi(z) \end{aligned} \quad (12)$$

The above equation suggests the only relevant quantity that determines BEC to take place is χ . In case of BEC as $T \rightarrow T_c$, $z \rightarrow 1$. So, BEC would take place when,

$$\chi = \frac{d}{l} + \sum_i^d \frac{1}{n_i} > 1 \quad (13)$$

And the critical temperature is,

$$T_c = \frac{1}{k} \left[\frac{N_c h^d b^{d/l} \prod_{i=1}^d c_i^{1/n_i}}{g C_n \Gamma(\frac{d}{l} + 1) V_d \prod_{i=1}^d \Gamma(\frac{1}{n_i} + 1) \zeta(\chi)} \right]^{\frac{1}{\chi}} \quad (14)$$

For massive bosons (with $l = 2$), one can find the BEC criterion,

$$\frac{d}{2} + \sum_i^d \frac{1}{n_i} > 1 \quad (15)$$

Therefore, for free massive bosons ($\sum_i^d \frac{1}{n_i} \rightarrow 0$), the above criterion reduces to

$$\frac{d}{2} > 1 \quad (16)$$

which coincides with the literature[5].

The ground state fraction [9, 14] from eq. (12) and (14),

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^\chi \quad (17)$$

The above equation produces the exact ground state fraction for free system when all $n_i \rightarrow \infty$ [5]. Again, obtaining

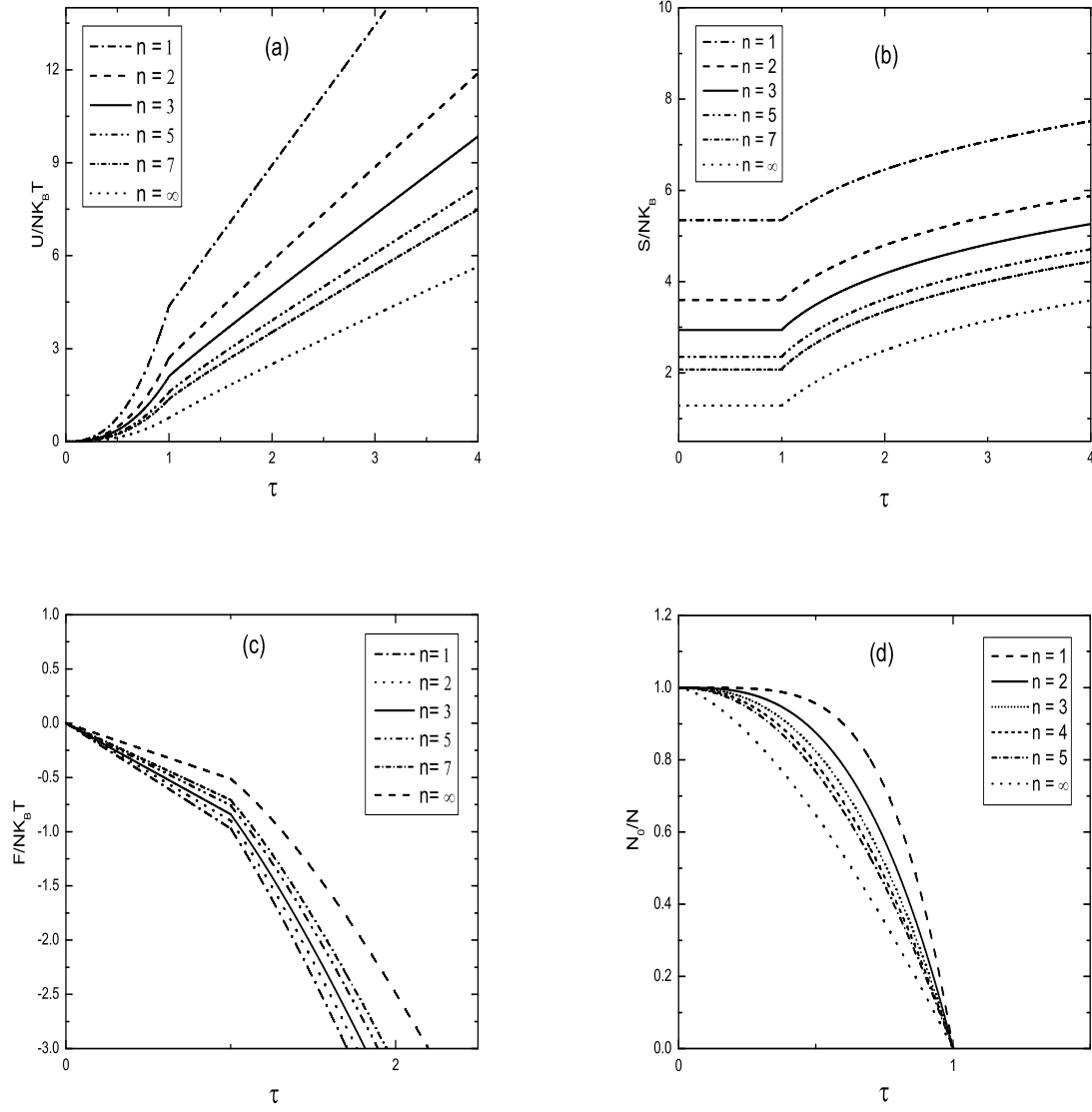


Figure 1: Fig (a), (b), (c) and (d) respectively represents internal energy, entropy, Helmholtz free energy and ground state fraction of Bose system, with different trapping potential in $d = 3$. $n = \infty$ denotes free Bose system.

internal energy E [9, 14] from the grand potential,

$$E = -\left(\frac{\partial q}{\partial \beta}\right)_{z,V} = \begin{cases} NkT\chi \frac{g_{\chi+1}(z)}{g_{\chi}(z)} & , T > T_c \\ NkT\chi \frac{\zeta(\chi+1)}{\zeta(\chi)} \left(\frac{T}{T_c}\right)^{\chi} & , T \leq T_c \end{cases} \quad (18)$$

Now, the entropy S [9, 14], below and greater than the critical temperature

$$S = kT\left(\frac{\partial q}{\partial T}\right)_{z,V} - Nk \ln z + kq = \begin{cases} Nk\left[\frac{v'_d}{\chi^d}(\chi+1)g_{\chi+1}(z) - \ln z\right] & , T > T_c \\ (N - N_0)k\frac{v'_d}{\chi^d}(\chi+1)\zeta(\chi+1) & , T \leq T_c \end{cases} \quad (19)$$

And, finally the expression of Helmholtz free energy,

$$\begin{aligned} A &= -kTq + NkT \ln z \\ \frac{A}{NkT} &= \begin{cases} -\frac{g_{\chi+1}(z)}{g_{\chi}(z)} + \ln z & , T > T_c \\ -\frac{\nu}{\chi^d}\zeta(\chi+1) & , T \leq T_c \end{cases} \end{aligned} \quad (20)$$

4 Heat capacity at constant volume C_V

Heat capacity at constant volume C_v below and above T_c [9, 14],

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_{N,V} = \begin{cases} Nk[\chi(\chi+1) \frac{g_{\chi+1}(z)}{g_{\chi}(z)} - \chi^2 \frac{g_{\chi}(z)}{g_{\chi-1}(z)}] & , T > T_c \\ Nk\chi(\chi+1) \frac{\zeta(\chi+1)}{\zeta(\chi)} \left(\frac{T}{T_c} \right)^{\chi} & , T \leq T_c \end{cases} \quad (21)$$

Another important quantity $\frac{\partial C_v}{\partial T}$ below and above T_C ,

$$\frac{1}{Nk} \frac{\partial C_v}{\partial T} = \begin{cases} \frac{1}{T} \left[\chi^2(\chi+1) \frac{g_{\chi+1}(z)}{g_{\chi}(z)} - \chi^2 \frac{g_{\chi}(z)}{g_{\chi-1}(z)} - \chi^3 \frac{g_{\chi}(z)^2 g_{\chi-2}(z)}{g_{\chi-1}(z)^3} \right] & , T > T_c \\ \frac{1}{T} \chi^2(\chi+1) \frac{\zeta(\chi+1)}{\zeta(\chi)} \left(\frac{T}{T_c} \right)^{\chi} & , T \leq T_c \end{cases} \quad (22)$$

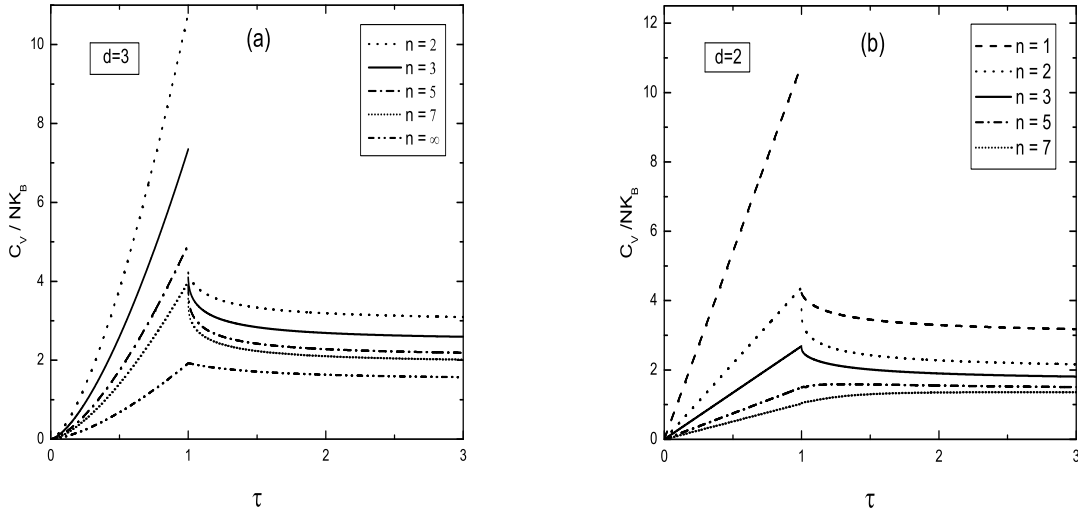


Figure 2: Fig (a) and (b), represents C_V of Bose system, with different trapping potential in $d = 3$ and $d = 2$, respectively.

Theorem 4.1: Let an ideal Bose gas in an external potential, $U = \sum_{i=1}^d c_i \left| \frac{x_i}{a_i} \right|^{n_i}$,

(i) if the Bose gas does condense ($\chi > 1$), heat capacity C_V will be discontinuous at $T = T_c$ if and only if,

$$\chi = \frac{d}{l} + \sum_{i=1}^d \frac{1}{n_i} > 2$$

(ii) And the difference between the heat capacities at constant volume, at T_c as

$$\Delta C_V \big|_{T=T_c} = C_V \big|_{T_c^-} - C_V \big|_{T_c^+} = Nk\chi^2 \frac{\zeta(\chi)}{\zeta(\chi-1)}$$

Proof:

As $T \rightarrow T_c$, $z \rightarrow 1$ and $\eta \rightarrow 0$, where $\eta = -\ln z$. For $T \rightarrow T_c^+$,

$$\begin{aligned} C_V(T_c^+) &= Nk[\chi(\chi+1)\frac{\nu'}{\lambda'^D}g_{\chi+1}(z)|_{z \rightarrow 1} - \chi^2 \frac{g_\chi(z)}{g_{\chi-1}(z)}|_{z \rightarrow 1}] \\ &= Nk[\chi(\chi+1)\frac{\nu'}{\lambda'^D}\zeta(\chi+1) - \chi^2 \frac{\zeta(\chi)}{g_{\chi-1}(z)}|_{z \rightarrow 1}] \end{aligned} \quad (23)$$

As the denominator of the second term of the right hand side contains $g_{\chi-1}(z)$, we can not simply substitute it with zeta function as $z \rightarrow 1$. So, using the representation of Bose function by Robinson (Eq. 11), we get from the above

$$C_V(T_c^+) = Nk[\chi(\chi+1)\frac{\nu'}{\lambda'^D}\zeta(\chi+1) - \chi^2 \frac{\zeta(\chi)}{\Gamma(2-\chi)}\eta^{2-\chi}|_{\eta \rightarrow 0}] \quad (24)$$

On the other hand

$$C_V(T_c^-) = Nk[\chi(\chi+1)\frac{\nu'}{\lambda'^D}\zeta(\chi+1)] \quad (25)$$

Taking the difference between $C_V(T_c^+)$ and $C_V(T_c^-)$, we get,

$$\Delta C_V|_{T=T_c} = \chi^2 \frac{\zeta(\chi)}{\Gamma(2-\chi)}\eta^{2-\chi}|_{\eta \rightarrow 0} \quad (26)$$

Which dictates, $C|_{T=T_c}$ will be non zero for $\chi > 2$. So, C_V will be discontinuous when $\chi > 2$ and thus completing first part of the theorem.

As, χ should be greater than two for ΔC_V at $T = T_c$ to be non-zero, we can re-write equation (23), by substituting $g_{\chi-1}(z)$ by zeta function.

$$C_V(T_c^+) = Nk[\chi(\chi+1)\frac{\nu'}{\lambda'^D}\zeta(\chi+1) - \chi^2 \frac{\zeta(\chi)}{\zeta(\chi-1)}] \quad (27)$$

Now, from Eq. (25) and (27) we can write.

$$\Delta C_V|_{T=T_c} = C_V|_{T_c^-} - C_V|_{T_c^+} = Nk\chi^2 \frac{\zeta(\chi)}{\zeta(\chi-1)}$$

In case of free Bose gas the first part of the theorem dictates there will be a jump in C_V at $T = T_c$ for $d > 4$, producing the same result as Ziff *et al.*

Theorem 4.2: Let an ideal Bose gas in an external potential, $U = \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i}$,

(i) the jump of the the first derivative of C_V at $T = T_c$ will be finite for $\chi = \frac{3}{2}$, it will be infinite for $\chi > \frac{3}{2}$ and no jump for $\chi < \frac{3}{2}$

(ii) And the finite jump of the first derivative of C_V , at $T = T_c$ is

$$\Delta \frac{\partial C_V}{\partial T}|_{T=T_c} = \frac{\partial C_V}{\partial T}|_{T_c^-} - \frac{\partial C_V}{\partial T}|_{T_c^+} = \frac{27Nk}{8T_c} [\zeta(\frac{3}{2})]^2 \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})}$$

Proof:

From equation (21) as $T \rightarrow T_c$, we obtain,

$$\frac{1}{Nk} \frac{\partial C_v}{\partial T} = \begin{cases} \frac{1}{T_c} \left[\chi^2(\chi+1) \frac{g_{\chi+1}(z)}{g_\chi(z)} - \chi^2 \frac{g_\chi(z)}{g_{\chi-1}(z)} - \chi^3 \frac{g_\chi(z)^2 g_{\chi-2}(z)}{g_{\chi-1}(z)^3} \right] & , T \rightarrow T_c^+ \\ \frac{1}{T_c} \chi^2(\chi+1) \frac{\zeta(\chi+1)}{\zeta(\chi)} & , T \rightarrow T_c^- \end{cases} \quad (28)$$

Now taking the difference,

$$\Delta \frac{\partial C_V}{\partial T} \big|_{T=T_c} = \frac{Nk}{T_c} \left[\chi^2 \frac{g_\chi(z)}{g_{\chi-1}(z)} + \chi^3 \frac{g_\chi(z)^2 g_{\chi-2}(z)}{g_{\chi-1}(z)^3} \right] \big|_{z \rightarrow 1} \quad (29)$$

Using the representation due to Robinson,

$$\Delta \frac{\partial C_V}{\partial T} \big|_{T=T_c} = \frac{Nk}{T_c} \left[\chi^2 \frac{\zeta(\chi)}{\Gamma(2-\chi)} \eta^{2-\chi} + \chi^3 \frac{\Gamma(3-\chi)}{[\Gamma(2-\chi)]^3} [\zeta(\chi)]^2 \eta^{3-2\chi} \right] \big|_{\eta \rightarrow 0} \quad (30)$$

Therefore the above equation suggests, $\Delta \frac{\partial C_V}{\partial T} \big|_{T=T_c}$ will be nonzero when $\chi \geq 3/2$ and zero elsewhere. Now, as $\chi > \frac{3}{2}$ the second term obviously diverges towards infinity and making the whole term infinite. But as $\chi = \frac{3}{2}$, the first term vanishes as $\eta \rightarrow 0$ and the second term becomes finite. Hence, completing the first part of the proof. Putting $\chi = \frac{3}{2}$ in Eq. (30) yields

$$\Delta \frac{\partial C_V}{\partial T} \big|_{T=T_c} = \left(\frac{3}{2}\right)^3 \frac{\Gamma(3-\frac{3}{2})}{[\Gamma(2-\frac{3}{2})]^3} [\zeta(\frac{3}{2})]^2 = \frac{27Nk}{8T_c} [\zeta(\frac{3}{2})]^2 \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} \quad (31)$$

Which completes the second part of the proof. In case of free massive (all $n_i \rightarrow \infty$) bosons in $d = 3$ Eq. (31) yields, the magnitude of the discontinuity of first derivative of C_V at $T = T_C$ is $3.665 \frac{Nk}{T_C}$, reproducing the exact same result of Pathria[5].

Theorem 4.3: Let an ideal Bose gas in an external potential, $U = \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i}$, The between l -th derivative of heat capacities at $T = T_C$ is,

$$\Delta^l(T_C) = \lim_{T \rightarrow T_C} [(\frac{\partial}{\partial T})^l C_v^-(T) - (\frac{\partial}{\partial T})^l C_v^+(T)] = \lim_{\eta \rightarrow 0} \sum_{j=1}^l a_{lj} \eta^{j+2-(j+1)\chi}$$

Proof:

We prove the above equation by the method of induction.

For $l = 1$,

$$\Delta^1(T_c) = \lim_{\eta \rightarrow 0} a_{11} \eta^{3-2\chi} \quad (32)$$

This is clearly true from equation (13) and the known result for $D = 3$.

Let's assume the equation holds for any positive integer l . Then,

$$\Delta^l(T_c) = \lim_{\eta \rightarrow 0} \sum_{j=1}^l a_{lj} \eta^{j+2-(j+1)\chi} \quad (33)$$

Now, in case of $l + 1$,

$$\begin{aligned} \Delta^{l+1}(T_c) &= \lim_{\eta \rightarrow 0} \sum_{j=1}^l a_{lj} (j+2-(j+1)\chi) \frac{\chi \zeta(\chi)}{T_c \Gamma(2-\chi)} \eta^{j+3-(j+2)\chi} \\ &= \lim_{\eta \rightarrow 0} \sum_{j=2}^{k+1} a_{l+1,j-1} (j+1-j\chi) \frac{\chi \zeta(\chi)}{T_c \Gamma(2-\chi)} \eta^{j+2-(j+1)\chi} \\ &= \lim_{\eta \rightarrow 0} \sum_{j=2}^{l+1} a_{l+1,j} \eta^{j+2-(j+1)\chi} \end{aligned} \quad (34)$$

a_{lj} satisfies the recurrence relation.

$$a_{l+1,j} = \frac{(j+1-j\chi)\chi\zeta(\chi)}{T_c \Gamma(2-\chi)} a_{l,j-1} \quad (35)$$

$$(36)$$

where, $j = 2, 3, 4, \dots, k + 1$.

Therefore, (a) and (b) enable us, for any positive integer l , to write

$$\Delta^l(T_c) = \lim_{\eta \rightarrow 0} \sum_{j=1}^l a_{lj} \eta^{j+2-(j+1)\chi} \quad (37)$$

This above equation coincides with Kim *et. al.* in case of free massive boson.

5 Latent Heat

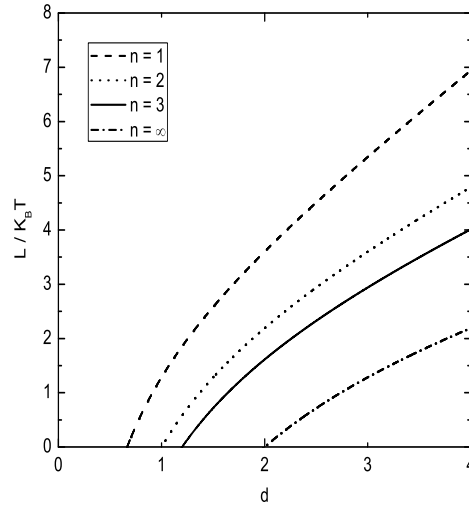


Figure 3: Latent heat of Bose condensation

Just as, any first order phase transition pressure is governed by Clausius-Clapeyron equation, in the transition line[6, 11]. Like the BEC for free Bose gas at $d = 3$, BEC for trapped Bose gas do also exhibit first order phase transition as they obey the Clausius-Clapeyron equation[6]. The Clausius-Clapeyron equation which is derived from Maxwell relations take the form,

$$\frac{dP}{dT} = \frac{\Delta s}{\Delta v} = \frac{L}{T \Delta v} \quad (38)$$

where, L , Δs and Δv are the latent heat, change in entropy and change in volume respectively. The effective pressure in phase transition line is,

$$P_0(T) = \frac{kT}{\lambda'^d} g_{\chi+1}(1) \quad (39)$$

Differentiating with respect to T leads,

$$\frac{dP_0}{dT} = \frac{1}{T v_g} [(\chi + 1) kT \frac{g_{\chi+1}(1)}{g_{\chi}(1)}] \quad (40)$$

When two phases coexist the non condensed phase has specific volume v_g whether the condensed phase has specific volume has specific volume 0, concluding $\Delta v = v_g$. So, the latent heat of transition per particle in case of trapped Boson is

$$L = (\chi + 1) \frac{\zeta(\chi + 1)}{\zeta(\chi)} kT \quad (41)$$

So, in the case of free massive boson the latent heat per particle in three dimensional space is,

$$L = \frac{5}{2}kT \frac{\zeta(5/2)}{\zeta(3/2)} \quad (42)$$

same as the text book results[6].

6 Discussion

In this section we discuss the influence of trapping potential on thermodynamic quantities. Beside this we also explain the theorems and their implications as well. In the figures, we have used $n_1 = n_2 = \dots = n_i = \dots = n_d = n$ (isotropic potential), but the formulas described in the above sections are more general.

In Fig (1), we have described the influence of different power law potentials on thermodynamic quantities such as internal energy E , entropy S , free energy F and ground state fraction of Bose gas in three dimensional space. In the case of internal energy, a strict nonlinear behavior is observed when $T < T_C$ and this behavior is more noticeable when we decrease the value of n . But in principle, the effect of power law potential is seen in both below and above T_C . Same phenomena is also observed in the case of S and F . Entropy remains constant in the condensed phase, with a specific choice of n . But the entropy increases while the value of free energy gets lower with trapping potential. Now turning our attention towards ground state fraction, Eq. (15) and (17) dictates, $|\frac{dN_0}{dT}|_{T=T_c} > \frac{N}{T_C}$. Thus, this relation depicts a very significant characteristic at the onset of BEC, that the $N_0 - T$ curves are always convex for Bose system in condensed phase. But their curvatures are different depending on trapping potentials as shown in figure 1 (d).

Fig 2 (a), (b) illustrates the C_V of Bose system with different power law potentials in three and two dimensional space, respectively. When $d = 3$, there is no discontinuity in free system for C_V at $T = T_C$. But according to theorem 4.1 in three dimensional space, C_V becomes discontinuous when $n < 6$, for isotropic trapping potential, which is visible in fig. 2 (a). In the case of $d = 2$, theorem A dictates that C_V becomes discontinuous when $n < 2$ for isotropic potential, also conspicuous in figure 2 (b). Therefore, we can conclude that theorem A can exactly govern the discontinuity condition of C_V at $T = T_C$. Now, let us turn our attention towards latent heat. Fig 3 illustrates the behavior of latent heat of trapped Bose gas, with changing dimensionality. As BEC is a 1st order phase transition[6], latent heat is associated with this phase transition. So, zero latent heat denotes no phase transition *i.e.* no condensed phase. As there is no phase transition in $d < 2$ for free Bose gas, there should be no latent heat associated with $d < 2$ in the case of free system, which is manifested in figure 3. But the scenario changes when we apply trapping potential. From Eq. (13) we can say, there will be condensed phase in $d = 2$ for any trapping potential unless $\sum_{i=1,2} \frac{1}{n_i} = 0$. Fig. 3 is in accordance with this fact as latent heat is seen to be non-zero with any trapping potential. In the case of $d = 1$ we get from Eq. (13) that BEC will take place if $n < 2$. It is seen in the Fig. 3 that latent heat is non-zero when $n = 1$ indicating the existence of condensed phase in one dimensional space.

Table 1: The hierarchy of the condensation transition of ideal Bose gas trapped under generic power law potential. The result of table are in agreement with Kim *et al.* for free system. But in the case of trapped system the "class" of functions significantly change depending on the values of n_i

χ	$(\partial/\partial T)C_v$	$(\partial/\partial T)^2C_v$	$(\partial/\partial T)^3C_v$	$(\partial/\partial T)^4C_v$	Class
$\chi = \frac{3}{2}$	d					C^0
$\frac{4}{3} \leq \chi < \frac{3}{2}$	c	d				C^1
$\frac{5}{4} \leq \chi < \frac{4}{3}$	c	c	d			C^2
\vdots						
\vdots						
$\frac{l+2}{l+1} \leq \chi < \frac{l+1}{l}$	c	c	c		...(d)	C^{l-1}
\vdots						
\vdots						
$\chi = 1$	c	c	c	c	c	C^∞

Let us now concentrate on the significance of theorem 4.3. The calculation done by Kim *et. al.* shows how the condensed phases can be different, depending on the dimensionality for free Bose gas. The dimensional dependence of discontinuity of the l th derivative of C_V indicates hidden hierarchy of the condensation transition with changing fractional dimensionality. Theorem 4.3 generalizes this result for trapped systems indicating a similar hierarchy where we find the class of l^{th} derivative of C_v depends on χ . We now elaborate how this theorem classifies the class of l^{th} derivative of C_v and present it in table 1.

(i) From Eq. (37), when $l = 1$, we see the difference between 1st derivative before and after T_c ,

$$\Delta = a_{11}\eta^{3-2\chi}|_{\eta \rightarrow 0}$$

In order to be discontinuous we need $\Delta \neq 0$, which will be true, when $3 - 2\chi \leq 0$ i.e. $\chi \geq \frac{3}{2}$. Furthermore, if $\chi > \frac{3}{2}$, the 1st derivative is infinitely discontinuous and $\chi = \frac{3}{2}$ denotes finite discontinuity.

(ii) Again, for $l = 2$,

$$\Delta = a_{21}\eta^{3-2\chi} + a_{22}\eta^{4-3\chi}|_{\eta \rightarrow 0}$$

Therefore, for the 2nd derivative to be discontinuous we need, $\frac{4}{3} \leq \chi < \frac{3}{2}$.

(iii) Similarly, for the l^{th} derivative to be discontinuous, the necessary condition is $\frac{l+2}{l+1} \leq \chi < \frac{l+1}{l}$

(iv) Careful observation of Eq. (37) reveals Δ^l is independent of j for $\chi = 1$, which indicates $\Delta = 0$ for $\eta \rightarrow 0$. Thus if $\chi = 1$ all derivatives of C_v are continuous. Using all these information, one can find out the hierarchy of the condensation transition (see table 1).

7 Appendix

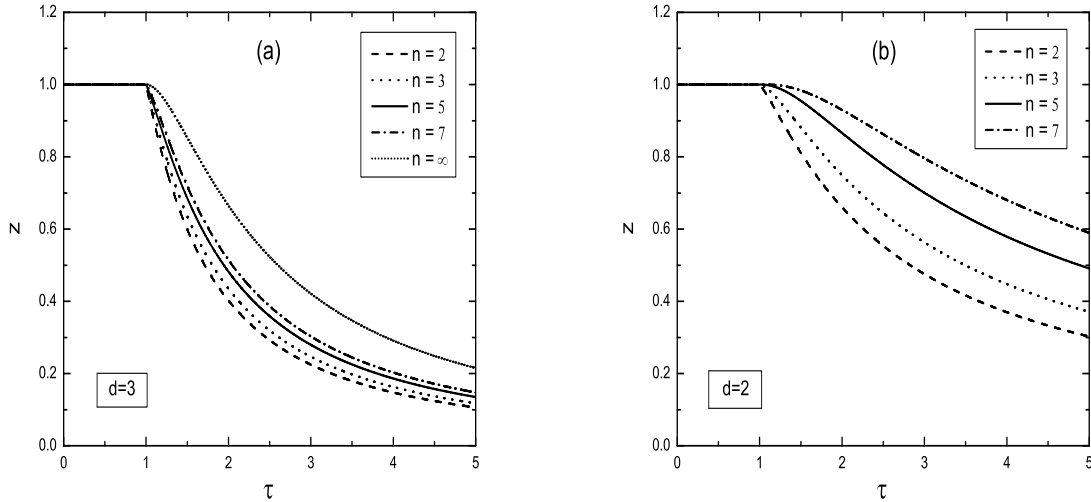


Figure 4: Fugacity as a function of $\tau = \frac{T}{T_c}$, with different power law potentials.

In this section we demonstrate how fugacity can be expressed as a function of $\tau = \frac{T}{T_c}$ following Kelly's work. The number of particles in the excited state near critical temperature,

$$N_e = \frac{1}{\lambda_c^d} \zeta(\chi)$$

And, the total number of particles can be written as,

$$N = \frac{1}{\lambda^d} g_\chi(z)$$

In the noncondensed phase, one can approximate $N_e \simeq N$. So, in that case we get from the above two equations,

$$\frac{\tau^{d/2} g_\chi(z)}{\zeta(\chi)} = 1$$

Solving the above equation numerically, using mathematica we get our desired result (see figure 4)

Another very important relation used in deriving the different thermodynamic quantities are,

$$\left(\frac{\partial z}{\partial T}\right)_V = -\chi \frac{z}{T} \frac{g_\chi(z)}{g_{\chi-1}(z)} \quad (43)$$

8 Conclusion

The changes in characteristics of Bose condensation of ideal Bose gas due to an external generic power law potential are studied from the grand potential. The presented theorems turn out to be very important for trapped Bose systems (non-relativistic). But it will be interesting to generalize these theorems for relativistic Bose gas.

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